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# Visualization of Complex Roots for Nonlinear Algebraic Equations

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**Abstract**

Visualization of complex roots of a nonlinear algebraic equation is discussed in this work. The method is based on calculating the modulus of the complex valued function and representing it as a surface in a three dimensional space where the axis consist of real and imaginary axis and the modulus function. Since it may be inconvenient to visualize multiple roots in a three dimensional surface, contour plot is suggested as an alternative to visualize better the location of roots. Roots of polynomial functions as well as non-polynomial functions are treated as examples. The contour plot is the best to visualize the complex roots in a single graph.

**Keywords**: Complex Roots, Algebraic Equations, Three Dimensional Surfaces, Contour Plots.

## Introduction

Determining the roots of algebraic equations is of technical importance in many branches of science. Searching for the natural frequencies of a vibrating system often ends up with a transcendental nonlinear algebraic equation for which the roots are the natural frequencies. Empirical equations are widely employed in fluid mechanics and heat transfer, which require determination of their roots. Excluding the very simple basic functions, one usually cannot manage to solve the roots algebraically and has to resort to numerical techniques. A vast literature exists on determining numerically the roots of nonlinear functions, the most common ones being interval halving, Newton-Raphson and Secant method. A general algorithm unifying the many single point iteration algorithms has been proposed as perturbation iteration algorithms (Pakdemirli & Boyacı, 2007).

In this discussion, the emphasis is on the visualization of locations of complex roots of nonlinear equations rather than their numerical calculations. One such technique is to plot a three dimensional modulus surface where the touching points of the surface to the two dimensional complex plane represents the location of the complex roots. Such surfaces were depicted for quadratic polynomial equations (Harding & Engelbrecht, 2007; Antino, 2009; Pall-Szabo, 2015) and for polynomial equations up to fourth order (William et al., 2018). A perspective view of the three dimensional modulus surfaces is incapable of visualizing all roots in a single graph since some of them may be hidden at the background. The surfaces may then be drawn for sub-regions each containing one of the roots. A better way to show the location of all the roots is to take contour plots of the surfaces. This enables to locate the roots in two dimensions in a single graph. It is obvious that if the nonlinear equation possesses infinitely many roots, then the visualization cannot be made in a single graph.

Another problem is the preliminary determination of the ranges of the real and imaginary axis without determining numerically the roots. For polynomials there are theorems developed which makes it enable to estimate the magnitudes of roots (Pakdemirli & Yurtsever, 2007; Pakdemirli & Elmas, 2010, Pakdemirli & Sarı, 2013, Pakdemirli et al. 2016). Such estimation of magnitudes of roots definitely gives an idea about the ranges of the axis. For non-linear functional equations where infinitely many roots may exist, there are no specific theorems to estimate the magnitudes of roots.

In this study, visualization of roots is depicted by the aid of three dimensional modulus surfaces and two dimensional contour lines. Two polynomial equations and two trancendantal equations are treated as examples. The contour lines form an aesthetic view also which can be used in math arts.

## Theory

The aim in this study is to visualize the roots of the nonlinear equation

$$f(x) = 0 \tag{1}$$

which may be real of complex. Assume that the root is expressed as

$$x_r = a_r + b_r i \tag{2}$$

with $i = \sqrt{-1}$ , $a_r, b_r \epsilon \boldsymbol{R}$. $f(a + bi)$ for general $a$ and $b$ values will produce a number with two components, one being real and the other being imaginary, i.e.

$$f(a + bi) = Re(f(a + bi)) + iIm(f(a + bi)) \tag{3}$$

Representation of $f(a + bi)$ in terms of $a$ and $b$ components would then be a four dimensional space which cannot be visualized. Instead, one may calculate the modulus of the function $f(a + bi)$

$$c = \sqrt{\left(Re(f)\right)^2 + \left(Im(f)\right)^2} \tag{4}$$

which is a real positive number. The parameters $a$, $b$ and $c$ can be expressed in a three dimensional space, $c$ representing the modulus surface of the complex valued function $f$. The modulus surface touches the two dimensional $a$, $b$ plane at the root locations corresponding to $c = 0$. The modulus $c$ is positive elsewhere from the definition of (4).

The perspective view of three dimensional surfaces may hide some of the roots if one tries to show all the roots in a single graph. A solution to this problem is to take contour lines of equal altitudes, i.e.

$$c = c_0 \tag{5}$$

to show the location of roots in a single two dimensional graph. In the vicinity of the roots, the contour lines will form loops inscribed within each other.

## Visualization of the Roots

Four example problems will be treated in this section, the first two being a polynomial function and the last two being transcendental equations.

## Polynomial Equations

**Example 1.**

Consider the polynomial equation

$$x^3 - x^2 + x - 1 = 0 \tag{6}$$

The first step is to select a range for real and imaginary components $a$ and $b$. From the first theorem given by Pakdemirli &Yurtsever (2007), it is proven that if all the coefficients of the polynomial are of the same order of magnitude, then the magnitudes of the roots are expected to be of order 1. Hence, the range for the parameters $a$ and $b$ can be selected as the interval [-2,2]. From the fundamental theorem of algebra, three roots should exist for the problem. They may all be real or one may be real and the remaining two roots being complex conjugates of each other.

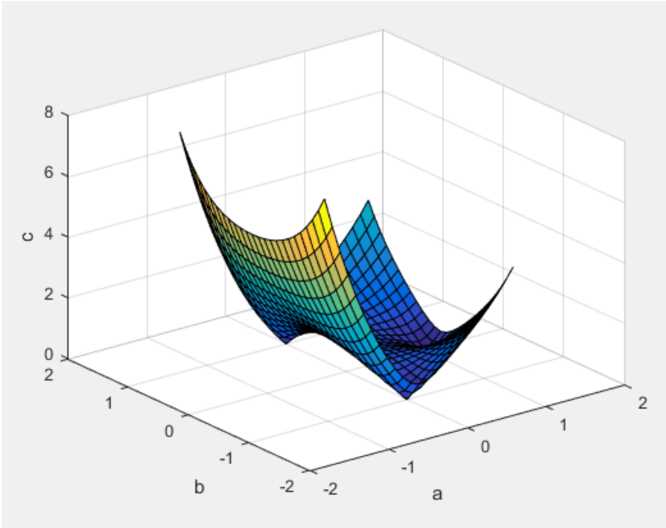


**Figure 1.** Modulus surface and roots of the polynomial equation (Example 1)

In Figure 1, the modulus surface is shown. The touch points on the ground represent the location of the roots which are 1, $+i$ and $-i$. The contour plots are given in Figure 2.

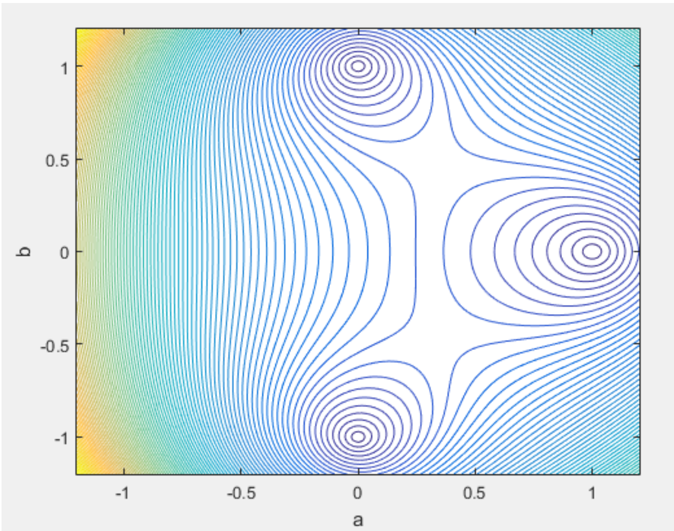


**Figure 2.** Contour curves of the polynomial equation (Example 1)

The centres of the inscribed loops represent the location of the roots. Although one can visualize all the roots in a single figure in Figure 1, it may not be possible in case of more roots. Figure 2 would then be a better choice to visualize all the roots. The MATLAB program which produces the figures is given in Table 1.

**Table 1.** Matlab Algorithm for Producing Modulus surfaces and contours (Example 1)

```
%Visualization of Complex Roots (Polinomial)
clear all
[a,b]=meshgrid(-1.2:0.01:1.2,-1.2:0.01:1.2);
x=a+b*i;
y=x.^3-x.^2+x-1;
cdiv=100;
c=(real(y).^2+imag(y).^2).^0.5;
%Plot of the modulus surface
surf(a,b,c)
cmin = floor(min(c(:)));
cmax = ceil(max(c(:)));
cinc = (cmax - cmin) / cdiv;
clevs = cmin:cinc:cmax;
```

```
%Plot of the contours
contour(a,b,c,clevs)
xlabel('a')
ylabel('b')
zlabel('c')
```

**Example 2.**

Consider the polynomial equation

$$x^5 + 0.08x^4 - 0.001x^3 + 0.01x^2 + 0.02x + 0.5 = 0 \quad (7)$$

The roots of this equation lie in the rectangular region [-1,1] for both axis (Pakdemirli & Sarı, 2013). From the fundamental theorem of algebra, five roots should exist for the problem, at least one of them being real.

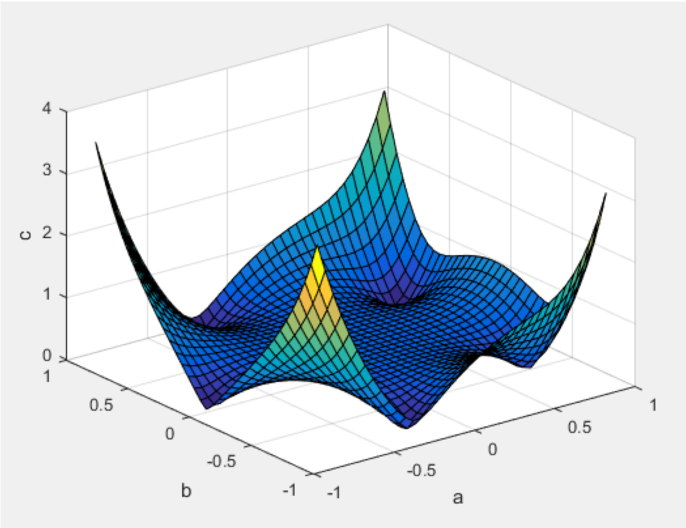


**Figure 3.** Modulus surface and roots of the polynomial equation (Example 2)

Figure 3 is a plot of the modulus surface, the touch points being the roots. To locate better the roots, the contour plot is given in Figure 4. The roots for the equation are $-0.8839$, $-0.2879 \mp 0.8220i$, $0.6899 \mp 0.5193i$ (Pakdemirli & Sari, 2013).

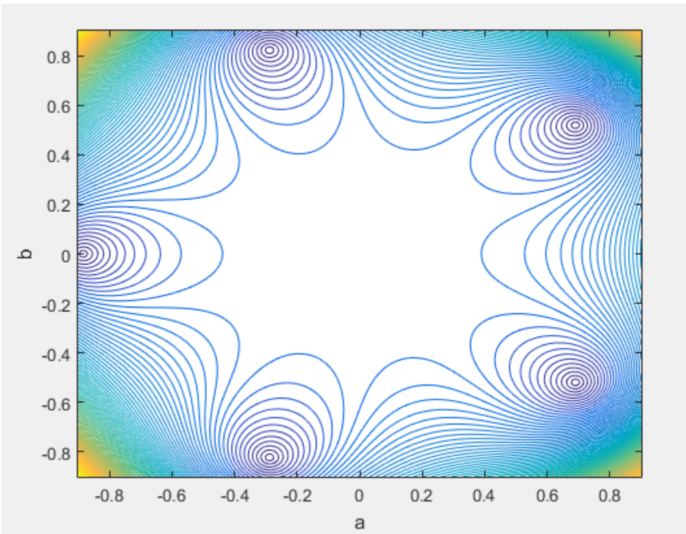


**Figure 4.** Contour curves of the polynomial equation (Example 2)

The centres of the inscribed loops represent the location of the roots**.** The location of the roots in the contour plot can be visualized better than the surface graph. Note that for polynomials, the complex roots are always symmetric with respect to *b*=0 axis and the real roots lay on this axis.

## Transcendental Equations

Two sample problems are treated in this section.

**Example 3.** Consider the nonlinear algebraic equation

$$e^x + 1 = 0 \quad (8)$$

No real roots exist for the equation since $e^x \geq 0$ always. Substituting $x = a + bi$ into (8) and separating real and imaginary parts with the aid of Euler formula ($e^{i\theta} = cos\theta + isin\theta$) and equating real and imaginary parts to zero yields

$$e^a cosb + 1 = 0, e^a sinb = 0 \quad (9)$$

the solution of which is $a = 0$, $b = \mp(2k + 1)\pi$, $k = 0,1,2, \ldots$. Therefore, there are infinitely many roots all being purely imaginary

$$x_r = \mp(2k + 1)\pi i, \qquad k = 0,1,2, \ldots \quad (10)$$

The first two positive roots are depicted in Figure 5.

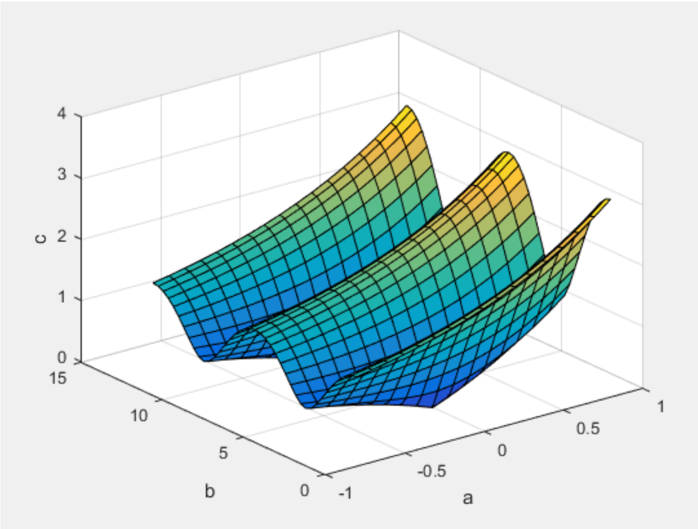


**Figure 5.** Modulus surface and roots of the nonlinear equation (Example 3)

The second root cannot be visualized in this perspective view. Contour plot gives a better view (Figure 6).

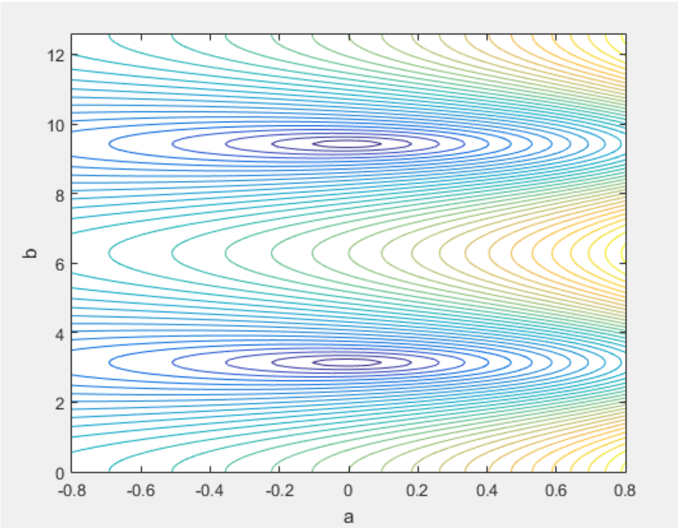


**Figure 6.** Contour curves of the functional equation (Example 3)

The centres of the inscribed loops represent the location of the roots.

**Example 4.** Consider the nonlinear algebraic equation

$$sinx + 2 = 0 \quad (11)$$

No real roots exist for the equation since $-1 \leq sinx \leq 1$.
The two roots in magnitude are shown as the touch points of the surface area in Figure 7.

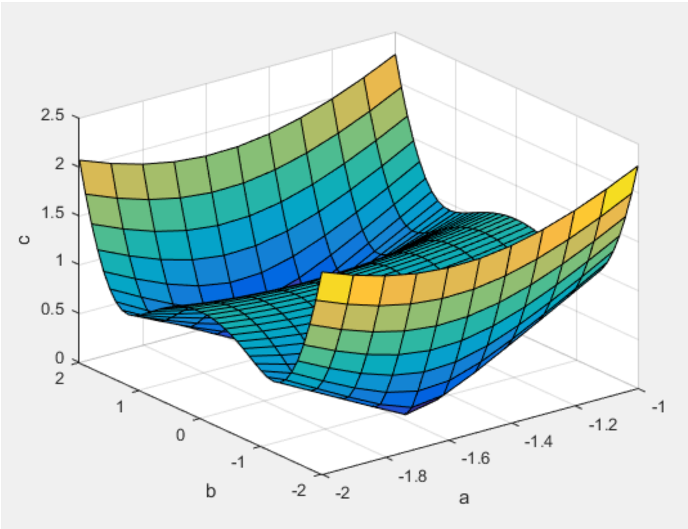


**Figure 7.** Modulus surface and two roots of the nonlinear equation (Example 4)

The obtain a better view, the contour plot is given in Figure 8.

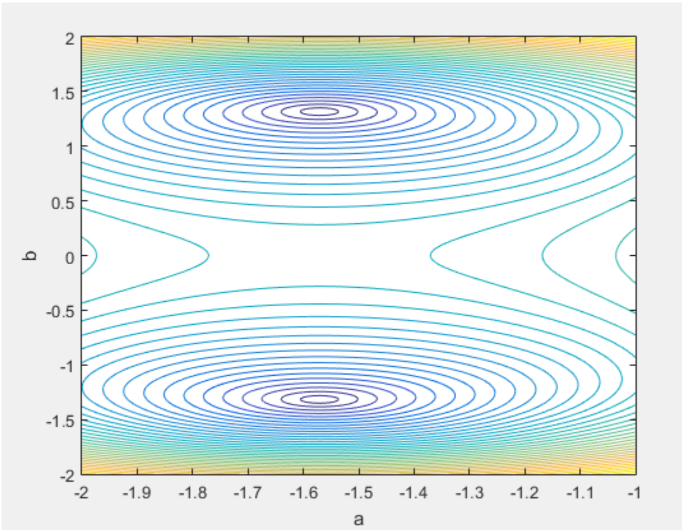


**Figure 8.** Contour curves of the functional equation (Example 4)

## Concluding Remarks

To visualize the roots of a nonlinear function, the modulus surfaces should be calculated as a first step. To better locate the roots, the contour curves of the surface may be drawn with the centres of inscribed loops determining the locations of the roots.

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