

Functional-Voxel Modeling of the Cauchy Problem

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Abstract

The paper considers an approach to solving the Cauchy problem for an example of a partial differential equation of the first order under given boundary conditions by the functional voxel method (FVM). The proposed approach uses the accumulated experience of differentiation and integration into FV- modeling to obtain local geometric characteristics of triangular elements on the surface of the resulting function in the process of linear approximation. The analytical solution of a simple example of a partial differential equation of the first order for the Cauchy problem is analyzed. Based on the obtained analytical solution, FV-model is constructed for further comparison with the results obtained by means of FV-modeling. The algorithm for solving the example is described by means of FV-modeling. A visual and numerical comparative analysis is carried out to determine the difference between the obtained results of FV-modeling and the accepted standard. The main difference between solving such a problem by numerical methods is the results obtained. In numerical methods, the result is the value of the function at the approximation nodes, and the FV-model at the nodes contains local geometric characteristics (gradient components in a space enlarged by one), which makes it possible to obtain a nodal local function of an implicit form, as well as a differential local function of an explicit form. The proposed graphical representation of the function area on a computer provides not only visual visibility, but also compact storage compared to a traditional array of real numbers.

Keywords: Functional-voxel modeling, partial differential equation, Cauchy problem, local function, local differential function, local geometric characteristics.

1. Introduction

The solution of a simple partial differential equation of the first order does not seem to be a difficult problem today and is fully provided by analytical and computer numerical solutions. The proposed study is aimed rather at obtaining a fundamental answer: is it possible to apply the properties of a functional voxel model (PV model) used to solve a fairly wide class of problems [1,2] in solving a differential equation in principle. Therefore, the example [3] was initially chosen, which is not complicated, but is able to demonstrate quite clearly the answer to the question posed. In the case of obtaining the resulting function manually or by means of an analytical calculator available on a computer, we have as a result a formula expression [4-8], but numerical computer methods are configured to obtain numerical values at the nodes of the approximation grid, i.e., we get as a result the number [9,10]. The functional voxel method (FV-method) provides filling on the specified area of the analytical function with local functions describing the local linear law for each point of the area, which makes it possible to apply in further calculations not just a number, but the corresponding analytical expression with all the advantages that follow from here.

In [11], the algorithm of differentiation and integration based on the principles of constructing the FV-model is considered. It is quite simple to switch to the FV model of partial derivatives

and back to the integrated FV model. At the same time, for the definiteness of the results obtained, it is necessary to set initial conditions (Cauchy problem). This allows us to develop an approach to the construction of a FV-model in solving a partial differential equation under given boundary conditions.

2. Problem statement

To demonstrate the algorithm, let's consider an example of solving a homogeneous partial differential equation, analyzed in the textbook [3]:

$$y \frac{\partial z}{\partial x} - x \frac{\partial z}{\partial y} = 0, \quad (1)$$

satisfying the condition

$$z(0, y) = y^2. \quad (2)$$

The analytical solution of such a differential equation will be a function describing a paraboloid of rotation $u(x, y)$:

$$z(x, y) = x^2 + y^2. \quad (3)$$

Herewith

$$\frac{\partial z}{\partial x} = 2x, \quad (4)$$

$$\frac{\partial z}{\partial y} = 2y. \quad (5)$$

Figure 1 shows an illustration of the function (3) defined on the domain $x \in [0, 2], y \in [0, 2]$, performed by the traditional approach in the *MathCAD* system with a discretization step of $1/30$.

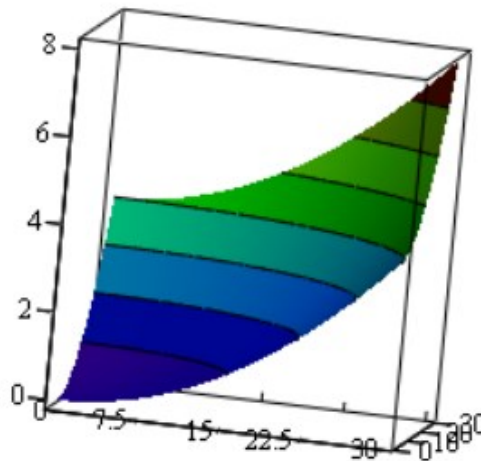


Fig. 1: Illustration of function (3) in the system *MathCAD*

Visual analysis of Figure 1 shows the approximate value of function (3) at the corner points of the surface segment under consideration, which, when accurately calculated, give coordinates: $(0, 0, 0)$, $(2, 0, 4)$, $(0, 2, 4)$, $(2, 2, 8)$.

Fig.2 (a, b) respectively demonstrates the image of the surfaces of functions (4) and (5) as partial derivatives of function (3). Let's calculate the values at the corner points for these surface segments, respectively: $(0, 0, 0)$, $(2, 0, 0)$, $(0, 2, 4)$, $(2, 2, 4)$ and $(0, 0, 0)$, $(0, 2, 0)$, $(2, 0, 4)$, $(2, 2, 4)$.

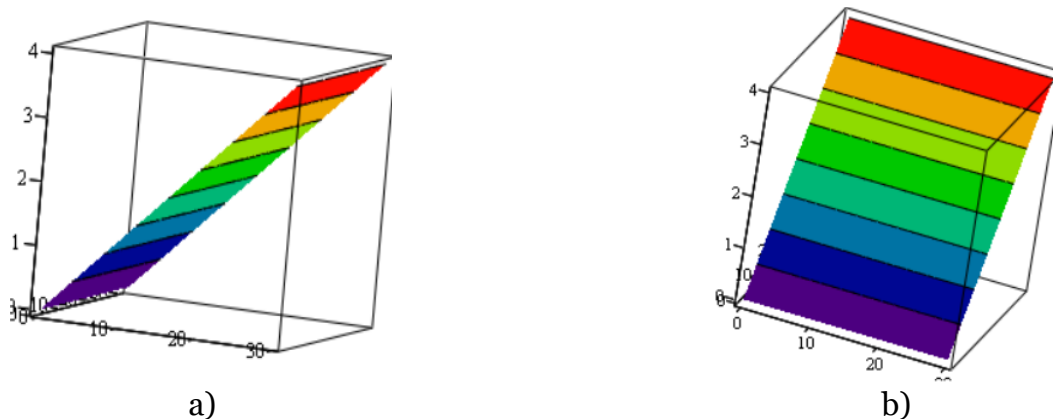


Fig. 2: Illustration of functions (4,5) in the system *MathCAD*

Further, applying the FV-method, we obtain a computer FV-representation of function (3) in the form of a domain of local functions of the form: $n_1x + n_2y + n_3z + n_4 = 0$, where (n_1, n_2, n_3, n_4) – the local geometric characteristics of such a function are represented on a computer in graphic form of *M*-images (M_1, M_2, M_3, M_4) (Fig.3) with a 400x400 image resolution. *M*-images in [8] are understood as model-images that display one of the local geometric characteristics of the FV-model in tone or color. Such a graphical representation of the function area on a computer provides not only visual visibility, but also compact storage compared to a traditional array of real numbers. At the same time, the accuracy of the representation of a numerical value by a semitone is provided in RGB format (256 shades). To increase visual clarity, we will demonstrate *M*-images for 16777214 shades of color in Figure 4. The patterns obtained in this case characterize the transition from shades of red through shades of green to blue, providing higher visibility due to the patterns provides for comparing the intended result with the proposed standard. In our case, we will take the *M*-images of Figures 3 and 4 as the standard of the solution.

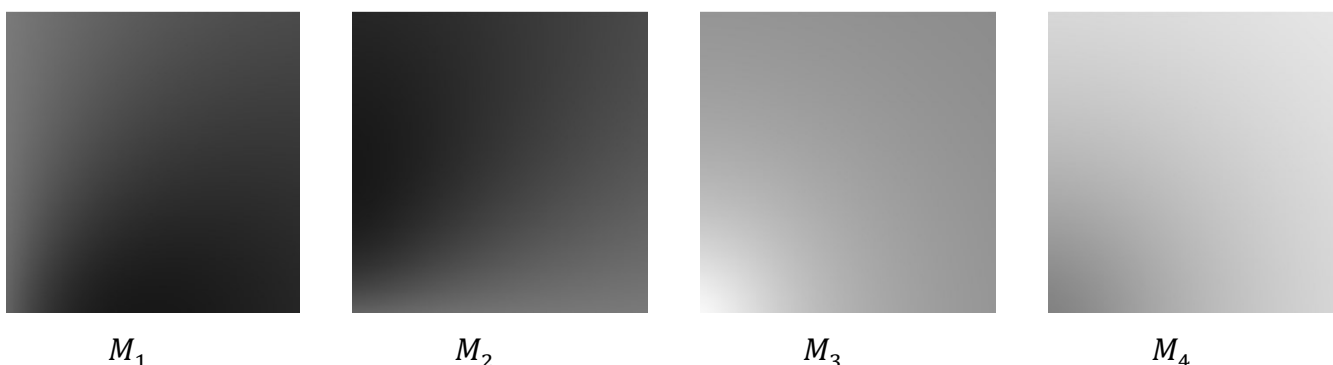


Fig. 3: Illustration of the representation on a computer of the local geometric characteristics of the function (3) (256 shades of semitone)

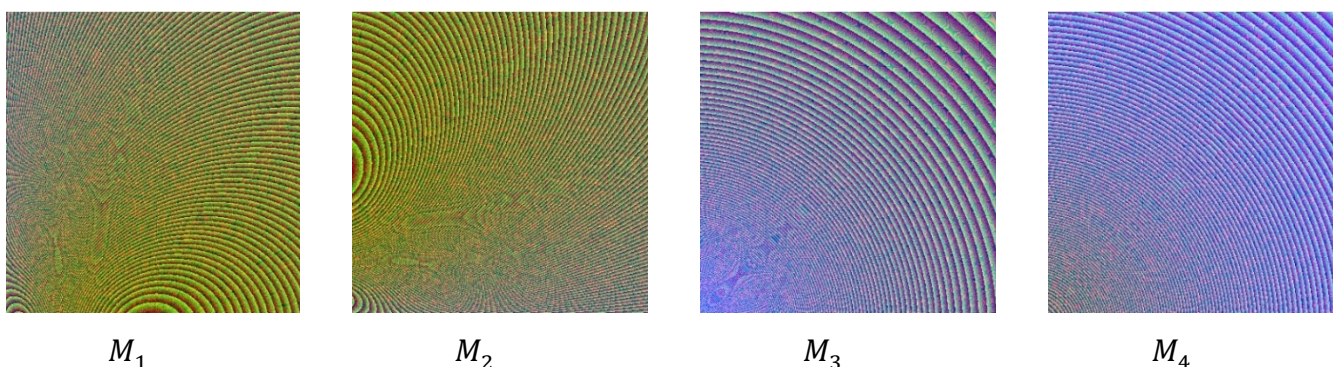


Fig. 4: Illustration of the representation on a computer of the local geometric characteristics of the function (3) (16777214 shades of color)

At this stage, we will assume that we have enough initial information for a numerical and visual experiment.

3. Construction of the FV-model of the differential equation

The paper [11] shows an algorithm for obtaining a FV-model of a integral function if its FV-models of partial derivatives are known. At the same time, it is sufficient to determine the local geometric characteristics at one approximation point to calculate the value of the function at the remaining points of the triangular element of the approximation grid. This makes it possible to expand the further search for local geometric characteristics over the entire given solution area.

To use this algorithm, it is necessary to express partial derivatives of functions to obtain their exact value at the current point. In the case under consideration, the initial condition is the function (2). It is a cross section of the desired surface of the function (3) when $x = 0$.

Which means

$$\frac{\partial z}{\partial y} \approx \frac{\Delta z}{\Delta y}, \quad \Delta y = h, \quad \Delta z = z_{i+1}(0, (i+1)h) - z_{i+1}(0, ih), i = [0 \dots n], \quad (6)$$

where h - approximation step.

Numerical data confirming the proposed solution can be seen in Figure 2, b. Here we see that at the point $(0, 0) \partial z / \partial y = 0$, and at the point $(400, 0) \partial z / \partial y = 0$. At the same time, along the Oy axis, the value is constantly increasing according to the linear law.

Then we can say that for $x=0$, a solution for determining partial derivatives is ready:

$$\frac{\partial z}{\partial x} \approx \frac{x \Delta z}{y \Delta y}. \quad (7)$$

If the local function of the FV-model is represented as: $ax + by + cu + d = 0$, then

$$\frac{\partial z}{\partial x} = -\frac{a}{c}, \quad \frac{\partial z}{\partial y} = -\frac{b}{c},$$

where the coefficient "c" is replaced by an approximation "C" (Рис.6):

$$C = x_1(y_2 - y_3) - x_2(y_1 - y_3) + x_3(y_1 - y_2)$$

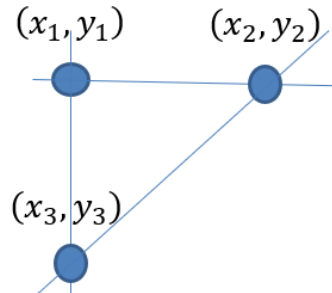


Fig. 6: Illustration of approximation nodes

Given the transition to obtaining the components of the gradient vector, we have:

$$A = -\frac{a}{c}C, \quad B = -\frac{b}{c}C, \quad D = -Ax_1 - By_1 - Cu_1. \quad (8)$$

At the first step of the calculation, u_1 is determined by the formula (3), and in other cases, respectively, the coefficients obtained through the next coefficients for the local function are used:

$$z_i = -\frac{A}{C}x_i - \frac{B}{C}y_i - \frac{D}{C}$$

As a result of sequential calculation for each node of the approximation triangulated grid of local geometric characteristics by the method of FV-modeling [6, 7], the area is filled of solution of the desired differential equation with local functions $n_1x + n_2y + n_3u + n_4 = 0$. On a computer, such an area is represented by the corresponding images (M_1, M_2, M_3, M_4) as shown in Fig.5 (256 gradations of halftone shading) and Fig.6 (16777215 gradations of RGB color shading).

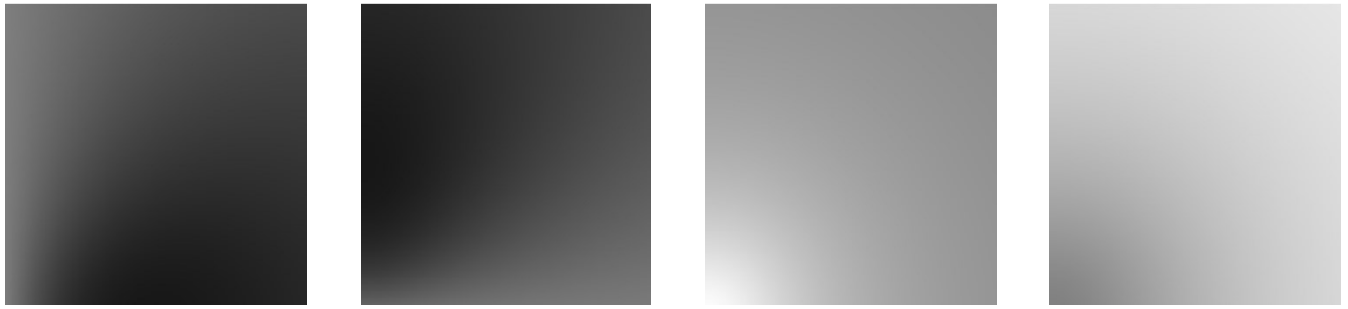


Fig. 5: Illustration of the local geometric characteristics of the obtained result (256 shades of semitone)

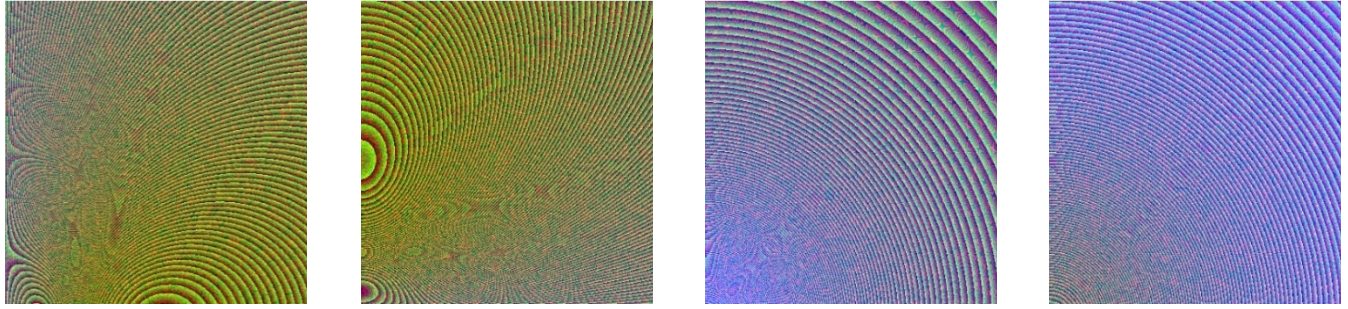


Рис. 6: Illustration of the local geometric characteristics of the obtained result (16777215 shades of color)

The result obtained in Fig.5, 6 is visually comparable with the result of Fig.3, 4. This confirms the adequacy of the algorithm.

Below is the result of numerical evaluation of the nodal values of the function and its partial derivatives at the corners of a given domain $x \in [0,2], y \in [0,2]$:

$x=0$	$y=0$	$z=0.0000$	$dz/dx=0.0000$	$dz/dy=0.0000$;
$x=0$	$y=2$	$z=3.9167$	$dz/dx=0.0000$	$dz/dy=3.9169$;
$x=2$	$y=0$	$z=3.9166$	$dz/dx=3.9453$	$dz/dy=0.0000$;
$x=2$	$y=2$	$z=7.9686$	$dz/dx=3.9453$	$dz/dy=3.9164$.

A comparison on the difference of points for the corresponding M-images with the accepted standards showed that for 640054 points of the image, the number of points with a value differing by no more than one:

$$M_1 = 9615, M_2 = 3554, M_3 = 2316, M_4 = 7186 \text{ (не более 1,9\%).}$$

4. Conclusions

The result of the obtained solution is a linear local function represented by local geometric characteristics for the points of the selected area: $n_1x + n_2y + n_3z + n_4 = 0$.

At the same time, expressing $u(x, y)$ we obtain a local differential equation:

$$z = -\frac{n_1}{n_3}x - \frac{n_2}{n_3}y - \frac{n_4}{n_3} \text{ or } z = \frac{\partial z}{\partial x}x + \frac{\partial z}{\partial y}y - \frac{n_4}{n_3}.$$

The presented studies are the initial stage in the application of FV-modeling in solving differential equations. Now we can say that the structure of the FV-model, based on information about the components of the gradient vector at each point of the function domain, discretely represents the differential characteristics of the function. But the question is how to use them effectively in solving a differential equation. To date, we can only say with confidence that without setting the necessary additional condition, the proposed approach will not be able to build the fully required FV-model.

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